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# Jet invariants of compact Lie groups

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**Abstract**

We solve, for irreducible representations, a conjecture proposed by D. Eck, analogous to the classical Hilbert theorem on polynomial invariants, relative to the $k$-jet action of a compact Lie group.


*Keywords:* Compact Lie groups; Jet bundles

## 1. Introduction

Let $M$ be a smooth manifold and let $m$ and $k$ be two fixed natural numbers. We denote by $M_k^m = J_0^k(\mathbb{R}^m, M) = \{j_0^k f : f \in C^\infty(\mathbb{R}^m, M)\}$ the manifold of $k$-jets at 0 of smooth maps from $\mathbb{R}^m$ to $M$. If $(y_i)$ are local coordinates in $M$, a set of local coordinates in $M_k^m$ is given by

$$y_i^{(\alpha)}(j_0^k f) = \frac{\partial^\alpha (y_i \circ f)}{\partial x^\alpha}(0), \quad \alpha = (i_1, \dots, i_m),\ |\alpha| \le k.$$

If $\varphi: M \to N$ is a smooth map, we define $\varphi_k^m : M_k^m \to N_k^m$ by $\varphi_k^m(j_0^k f) = j_0^k(\varphi f)$.

Let $G$ be a Lie group with Lie algebra $\mathcal{G}$; then $G_k^m$ and $\mathcal{G}_k^m$ inherit in a natural way a Lie group and algebra structure respectively. If $\exp: \mathcal{G} \longrightarrow G$ is the exponential map, then the induced map $\exp_k^m: \mathcal{G}_k^m \longrightarrow G_k^m$ is just the exponential map between a Lie algebra and its Lie group. Eck details in [1] that this functorial property in essence is due to the fact that if $\tau(t)$ is a smooth curve in $G$ which depends also polynomially in $m$ variables $\{x_1, \dots, x_m\}$ up to degree $k$, the same thing happens with the corresponding tangent vector field, and hence each one-parameter group in $G_k^m$ has by the infinitesimal generator an element $\mathcal{G}_k^m$.

Let $V$ be a real vector space on which $G$ acts linearly, and let us denote by $\mathbf{P}[V]$ the ring of polynomial functions on $V$; given $p \in \mathbf{P}[V]^G$, then $p_k^m : V_k^m \to \mathbb{R}_k^m$ defines a $G_k^m$-invariant polynomial mapping whose $\alpha$-component function, given by $p^{(\alpha)}(j_0^k f) = \partial^\alpha (p \circ f)/\partial x^\alpha(0)$, is an element of $\mathbf{P}[V_k^m]^{G_k^m}$. Eck wonders in [1] for what groups $G$ and $G$-modules $V$ is it true that the set

$$\left(\mathbf{P}[V]^G\right)_k^m = \{p^{(\alpha)} : p \in \mathbf{P}[V]^G, |\alpha| \le k\},$$

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which we shall call the ring of *Eck polynomials*, generates $\mathbf{P}[V_k^m]^{G_k^m}$. In [1] Eck solves the problem in the case where $G = SO(n)$ operates in the canonical way in $V = \mathbb{R}^n$, proving that since $\mathbf{P}[V]^G = \mathbf{P}[\theta]$, where $\theta[v_1, \ldots, v_n] = \sum v_i^2$, then $\mathbf{P}[V_k^m]^{G_k^m}$ is generated by $\{\theta^{(\alpha)}, |\alpha| \leq k\}$. His method is rather complicated and not intrinsic and although he hopes that the technique developed in this particular case might shed light on the general case, when he approaches the matter subsequently [2] he studies the problem in the framework of complex algebraic varieties, to prove, in particular, that if $G$ is a compact Lie group then the ring $\mathbf{P}[V_k^m]^{G_k^m}$ is a Nagata localization of $\left(\mathbf{P}[V]^G\right)_k^m$.

In this article we shall prove that if $G$ is a connected compact Lie group provided with an finite irreducible representation in $V$, then $\mathbf{P}[V_k^m]^{G_k^m} = \left(\mathbf{P}[V]^G\right)_k^m$.

## 2. Algebraic invariants of *k*- jet actions

Let $G$ be a Lie group, $\rho: G \times V \longrightarrow V$ a finite-dimensional representation of $G$, and $\rho_k^m$ the induced representation of the group $G_k^m$ on $V_k^m$. Each element $A \in \mathcal{G}$ induces a vector field $A^* \in \mathcal{X}(V)$ called the fundamental vector field associated to $A$, generated by the one-parameter subgroup $a_t = \exp tA$ on $V$. The set of fundamental vector fields $L_G$ has a Lie algebra structure [3] and will be called the *fundamental algebra* associated to the action of $G$ on $V$.

Let us denote by $S^r(V)$ the $r$-th symmetric power of $V$ and define $S^\circ(V) = \bigoplus_{r\geq 0} S^r(V)$. A vector field $X \in \mathcal{X}(V)$ is said to be algebraic if it stabilizes the subalgebra $S^\circ(V^*) \subset C^\infty(V)$. Moreover $X$ is said to be homogeneous of degree $r$ if $X(S^r(V^*)) \subseteq S^{k+r-1}(V^*)$. The infinitesimal generator of the group of homotheties $\varphi_t: V \longrightarrow V, \varphi_t(e) = \exp(t)e$ is called the Liouville vector field of $\mathcal{L}$ on $V$, and it is not hard to see that a homogeneous vector field $X \in \mathcal{X}(V)$ is of degree $d$ if and only if $[\mathcal{L}, X] = (d-1)X$. In this way, since the automorphisms of $V$ commute with homotheties, the fundamental algebra $L_G$ of a Lie group $G$ is a homogeneous algebraic distribution of degree 1.

**Definition 1.** An algebraic distribution $\mathcal{D}$ on $\mathbb{R}^n$ is called faithfully algebraic if for each open set $U$ the ring of sections $\Gamma(U, E_{\mathcal{D}})$ of the sheaf of first integrals $E_{\mathcal{D}}$ of $\mathcal{D}$ on $U$ is generated by polynomial functions.

We do not know necessary and sufficient conditions on a Lie group $G$ such that the fundamental algebra associated to one representation will be faithfully algebraic. In the case that $G$ is a compact Lie group, Hilbert's theorem asserts that $P(\mathbb{R}^n)^G$ is finitely generated as an $\mathbb{R}$-algebra (cf. [7]). Moreover taking into account that if $X_1, \ldots, X_r$ span $L_G(V)$ then they also span $L_G(W)$ for each open subset $W \subset V$ (which is an easy consequence of the fact that $L_G$ is a quasi-flasque sheaf (cf. [6]), we have:

**Proposition 2.** *Let $G$ be a connected compact Lie group and $\rho$ a representation in $V = \mathbb{R}^n$; then the associated fundamental algebra $L_G$ is a faithfully algebraic distribution.*

**Proposition 3.** *Let $G$ be a connected Lie group acting on $V = \mathbb{R}^n$, such that $L_G$ is a faithfully algebraic distribution. If $r = \max_{v\in\mathbf{R}^n}(rk(L_G)_v)$ then the ring $\Gamma(\mathbb{R}^n, E_{L_G})$ has $n-r$ functionally independent generators.*

**Proof.** We set $C = \{v \in \mathbb{R}^n : rk(L_G)_v < r\}$; this is a closed algebraic submanifold of $\mathbb{R}^n$ and then its complementary open $U$ is dense in $\mathbb{R}^n$. In this way $L_G|U$ is a regular distribution of vector fields of rank $r$ and hence by the Frobenius theorem admits a maximal system of $n-r$ functionally independent first integrals, which we suppose polynomial functions on $U$. Now, it is clear that if $X \in \Gamma(\mathbb{R}^n, L_G)$ and $f \in C^\infty(p_1, \ldots, p_{n-r})$, then $X(f)$ vanish identically by vanishing on an open dense subset. □

Thus, considering the Frobenius distribution which provides the fundamental algebra $L_G$ associated to the action of $G$ on $\mathbb{R}^n$, we give an easy proof of a fundamental theorem on differentiable invariants under the action of a compact Lie group $G$. If $\{p_1, \ldots, p_l\}$ is a minimal Hilbert basis, then we write $p = (p_1, \ldots, p_l): \mathbb{R}^n \longrightarrow \mathbb{R}^l$. The map $p^* : P(\mathbb{R}^l) \to P(\mathbb{R}^n)^G$ defined by $p^*(q) = q \circ p$ is surjective by Hilbert's theorem. Now, let us denote by $C^\infty(\mathbb{R}^n)$ the ring of real valued $C^\infty$ functions on $\mathbb{R}^n$; we have:

**Corollary 4** (*Schwarz [5]–Mather [4]*). *The induced mapping*

$$p^* : C^\infty(\mathbb{R}^l) \to C^\infty(\mathbb{R}^n)^G.$$

*is surjective.*

**Theorem 5.** *Let G be a connected compact Lie group provided with an irreducible linear action in* $V = \mathbb{R}^n$. *Then the ring of algebraic invariants of the action of* $G_k^m$ *in* $V_k^m$ *is given by*

$$\mathbf{P}[V_k^m]^{G_k^m} = (\mathbf{P}[V]^G)_k^m.$$

**Proof.** It is clear that $\dim_{\mathbf{R}} V_k^m = n\beta_k^m$ where $\beta_k^m = \binom{m+k}{k}$. Let $U \subseteq V$ be the open dense subset where the distribution $L_G$ has maximal rank $r$ and where we can choose $n-r$ polynomial functions $\{p_i, |i| \leq n-r\}$ that are first integrals of $L_G$ generating $\mathbf{P}[V]^G$. Let $U_k^m = J_0^k(\mathbb{R}^m, U)$ be the $k$-jet manifold of $U$; then $L_{G_k^m}|U_k^m$ has by rank $r\beta_k^m$, and by the Frobenius theorem the ring of first integrals is generated by $n\beta_k^m - r\beta_k^m$ functions. Now, the cardinal of the set of the Eck polynomials $\{p_i^{(\alpha)}, |i| \leq n-r, |\alpha| \leq k\}$ is $(n-r)\beta_k^m$ and hence

$$\Gamma\left(U_k^m, E_{L_{G_k^m}}\right) = \left\langle p_i^{(\alpha)}, |i| \leq n-r, |\alpha| \leq k \right\rangle.$$

To conclude we need to prove that every polynomial system of generators of $\Gamma(V, E_{L_G})$ also generates the ring $\mathbf{P}[V] \cap \Gamma(V, E_{L_G})$ and that the same thing happens with a system of generators of $\Gamma(V_k^m, E_{L_{G_k^m}})$ with respect to the ring $\mathbf{P}[V_k^m] \cap \Gamma(V_k^m, E_{L_{G_k^m}})$.

We shall say that a submanifold $W$ of $R^n$ is $G$-stable if for each point $p \in W$ the whole orbit $G \cdot p$ lies on $W$.

In this way, we shall proceed by induction on the dimension of the $G$-stable submanifolds of $\mathbb{R}^n$. We shall say that $G$ operates irreducibly on a $G$-stable $W \subseteq \mathbb{R}^n$ submanifold if each $G$-orbit lying on $W$ is connected. Since $G$ operates in a continuous way in $\mathbb{R}^n$, the irreducibility of this action implies the irreducibility of the action in such a $G$-stable submanifold of $\mathbb{R}^n$.

Thus, let $\{p_1, \ldots, p_l\}$ be a set of functionally independent polynomials generating $\Gamma(V, E_{L_G})$ and let $p = (p_1, \ldots, p_l)$ denote the corresponding map from $\mathbb{R}^n$ to $\mathbb{R}^l$. We let $S^{l-1}$ denote the unit sphere in $\mathbb{R}^l$ and we let $W = p^{-1}(S^{l-1})$. Then $W$ is a compact $G$-stable submanifold of $\mathbb{R}^n$ of codimension 1. In order to see this, we can suppose that $G$ acts orthogonally on $\mathbb{R}^n$ (if the standard metric $T_2$ in $\mathbb{R}^n$ is not $G$-orthogonal, we take the new metric $T_2' = \int_G g^*(T_2)\mathrm{d}g$, where $\mathrm{d}g$ is the Haar measure on $G$) hence the square of the radius function on $\mathbb{R}^n$ is a proper map and it is a polynomial in the elements of the set $\{p_1, \ldots, p_l\}$; hence $p$ itself is a proper map. Now, $x \in W$ if and only if $f(x) = \sum_{i=1}^l p_i^2(x) - 1 = 0$, and in this way $W$ has codimension 1 and the $G$-invariance of $f$ implies the $G$-stability of $W$.

Given $q \in \mathbf{P}[W]^G$, let us denote by $\overline{q}$ its restriction to $W$. If we consider the quotient map $\pi : \mathbb{R}[x_1, \ldots, x_n] \longrightarrow \mathbb{R}[x_1, \ldots, x_n]/(p_1^2 + \cdots + p_l^2 - 1)$ then $\overline{q} = \pi(q)$. The restriction of $L_G$ to $W$ has $\{\overline{p}_1, \ldots, \overline{p}_{l-1}\}$ as a system of generators of the ring of global sections of the sheaf of first integrals $E_{\overline{L_G}}$, and in this way there exists a polynomial $Q$ such that $\overline{q} = \overline{Q}(\overline{p}_1, \ldots, \overline{p}_{l-1})$, and as $q \equiv Q(p_1, \ldots, p_{l-1}) \mathrm{mod}(p_1^2 + \cdots + p_l^2 - 1)$, we conclude the first part of the proposition.

In the same way, the ring of first integrals of the restriction of $E_{L_{G_k^m}}$ to $W_k^m$ is generated by the projection of $\{p_i^{(\alpha)}, |i| \leq l-1, |\alpha| \leq k\}$ to the coordinate ring of $W_k^m$. Thus if $P \in \mathbf{P}[V_k^m]^{G_k^m}$, then there exists a polynomial $R$ such that the restriction of $P$ to $W_k^m$ is of the form $\overline{P} = \overline{R}(\overline{p}_i^{(\alpha)})$ ($|i| \leq l-1$). But now $P \equiv R(p_i^\alpha) \mathrm{mod}\, I(W_k^m)$, and as the ideal of $W_k^m$ in $V_k^m$ is generated in the form $I(W_k^m) = \langle P_1 = p_1^2 + \cdots + p_l^2 - 1, P_1^{(\alpha)}, |\alpha| \leq k\rangle$ by [2], we conclude that $P$ is a polynomial in the Eck polynomials. This completes the proof of the theorem. □

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